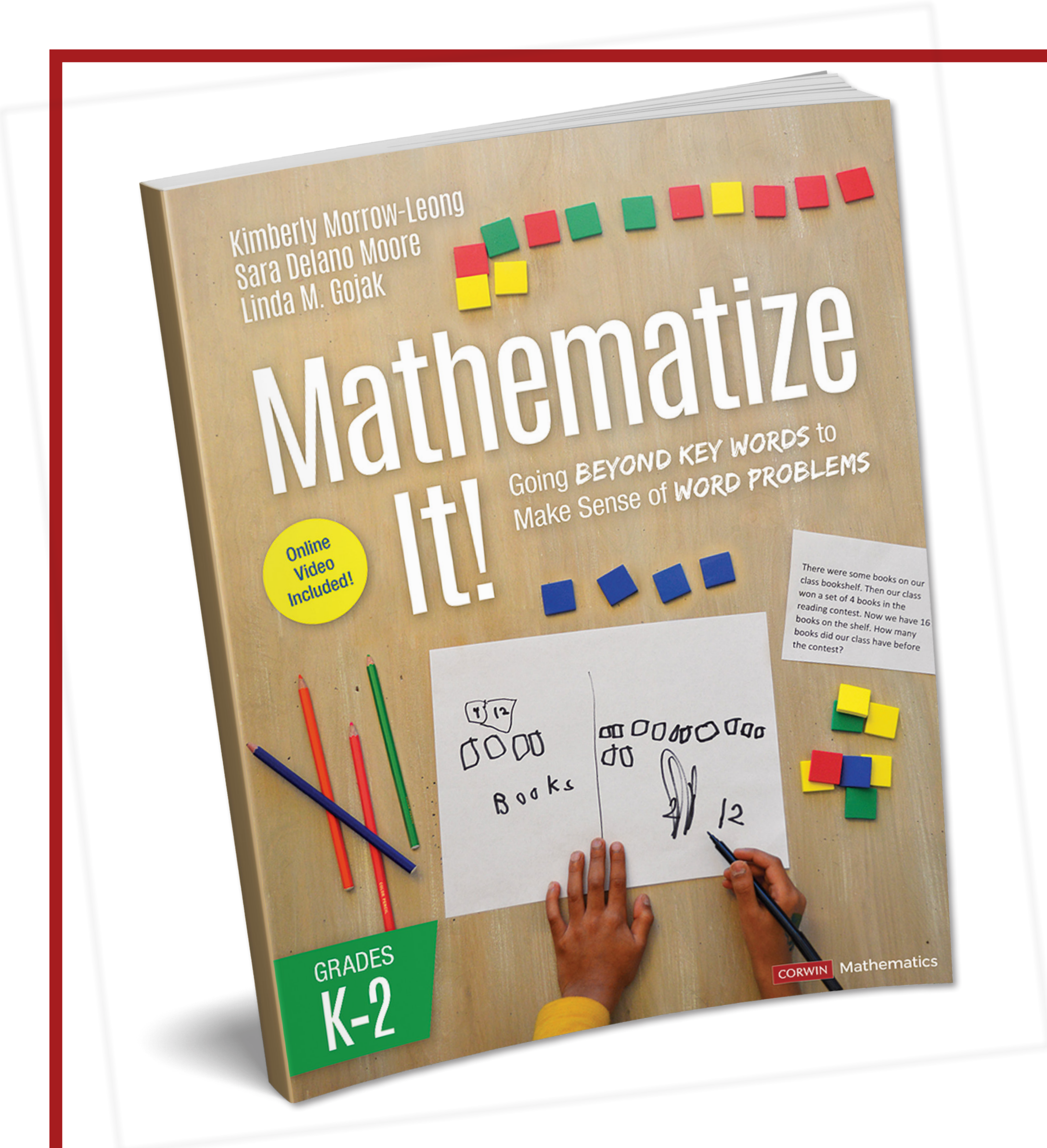


Thank you

FOR YOUR INTEREST IN CORWIN

Please enjoy this complimentary excerpt from *Mathematize It!, Grades K-2*. Chapter two explains how to move from counting to adding and subracting.

LEARN MORE about this title, including Features, Table of Contents, and Reviews.

CHAPTER TWO

Moving From Counting to Addition and Subtraction

Thinking About Counting, Addition, and Subtraction

In this chapter we introduce the principles of early counting and make connections to the work students first do with the operations of addition and subtraction. You might think of addition and subtraction as being fairly straightforward operations, but as you'll see in this book, there is some nuance in the way children make sense of these operations that is important to keep in mind. In this chapter we will talk about two kinds of *active situations*—meaning that something is moving within the problem, that is, something is being added to or taken away. We refer to these as Add-To and Take-From situations. We will focus on the Result Unknown variation of these problems in order to compare the action with counting. We'll get into other kinds of situations in later chapters. Let's get started.

Pretend you are walking into a workshop. This is the first of six workshops exploring problem situations that all students will encounter in grades K through 2. In this workshop, as in the other five, you will be exploring in detail aspects of these problems and the operations associated with them that you may not have considered before. Doing so requires that you take on the role of student, see the problems with new eyes, and let yourself try out representations and models for yourself.

Addition and Subtraction Problem Situations

	Result Unknown
Add-To	Paulo counted 9 crayons. He put them in the basket. Paulo found 6 more crayons under the table. He put them in the basket. How many crayons are in the basket? $9+6=x$ $6=x-9$
Take-From	There are 19 students in Mrs. Amadi's class. 4 students went to the office to say the Pledge. How many students are in the class now? $19-4=x$ $4+x=19$

Note: The representations for the problem situations in this table reflect our understanding based on a number of resources. These include the tables in the Common Core State Standards for Mathematics (CCSS-M; National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), the problem situations as described in the Cognitively Guided Instruction research (Carpenter, Hiebert, & Moser, 1981), and other tools. See the Appendix and the book's companion website for a more detailed summary of the documents that informed our development of this table.

ACTIVE SITUATIONS				
	Result Unknown	Change Addend Unknown	Start Addend Unknown	
Add-To	Paulo counted 9 crayons. He put them in the basket. Paulo found 6 more crayons under the table. He put them in the basket. How many crayons are in the basket? $9+6=x$ $6=x-9$	Paulo counted 9 crayons. He found more and put them in the basket. Now Paulo has 15 crayons. How many crayons did he put in the basket? $9+x=15$ $9=15-x$	Paulo had some crayons. He found 6 more crayons under the table. Now he has 15 crayons. How many crayons did Paulo have in the beginning? $x+6=15$ $15-6=x$	
Take-From	There are 19 students in Mrs. Amadi's class. 4 students went to the office to say the Pledge. How many students are in the class now? $19-4=x$ $4+x=19$	There are 19 students in Mrs. Amadi's class. Some students went to class to read the Pledge. There were still 15 students in the classroom. How many students went to the office? $19-x=15$ $x+15=19$	4 students went to the office. 15 students were still in the classroom. How many students are there in Mrs. Amadi's class? $x-4=15$ $15+4=x$	
RELATIONSHIP (NONACTIVE) SITUATIONS				
	Total Unknown	One Part Unknown		Both Parts Unknown
Part-Part-Whole	The first grade voted on a game for recess. 11 students voted to play four square. 8 voted to go to the playground. How many students are in the class? $8+11=x$ $x-11=8$	The 19 first graders voted on a recess activity. 8 students voted to go to the playground. How many wanted to play four square? $8+x=19$ $x=19-8$		The 19 first graders voted on a recess activity. Some wanted to play four square. Some wanted to go to the playground. What are some ways the first graders could have voted? $x+y=19$ $19-x=y$
	Difference Unknown	Greater Quantity Unknown	Lesser Quantity Unknown	
Additive Comparison	Jessie's paper airplane flew 14 feet. Jo's paper airplane flew 9 feet. How much less did Jo's paper airplane fly than Jessie's? $14-9=x$ $9+x=14$	Jo's paper airplane flew 9 feet. Jessie's paper airplane flew 5 feet more than Jo's. How far did Jessie's paper airplane fly? $9+5=x$ $x-5=9$	Jessie's paper airplane flew 14 feet. Jo's paper airplane flew 5 feet less than Jessie's paper airplane. How far did Jo's paper airplane fly? $14-5=x$ $14=x+5$	

Sandbox Notes: Explore Your Thinking

As you enter into problem exploration mode, gather your tools, including markers or crayons, linking cubes, base 10 blocks, linking cubes, counters, and any other tools that you routinely have available in your classroom. Try several of the concrete manipulatives and some hand-drawn picture models that reflect the **mathematical story** in the word problem. If you put the problem in your own words, revisit that rephrasing and specify where you can see each quantity in the problem. Where can you see each quantity in the models you have created? Think about how your work can express your understanding of the problem situation.

Ask yourself these questions to focus your thinking:

- Think about the quantities in each situation. What do they represent? What action is taking place between the quantities in the problem?
- How can you represent the quantities in the word problems with your manipulatives or pictures?
- How can you make a number sentence about the problem?

Mathematical story: A retelling of the actions or relationships in a word problem or other problem context in a way that highlights the important mathematical details over any other information.

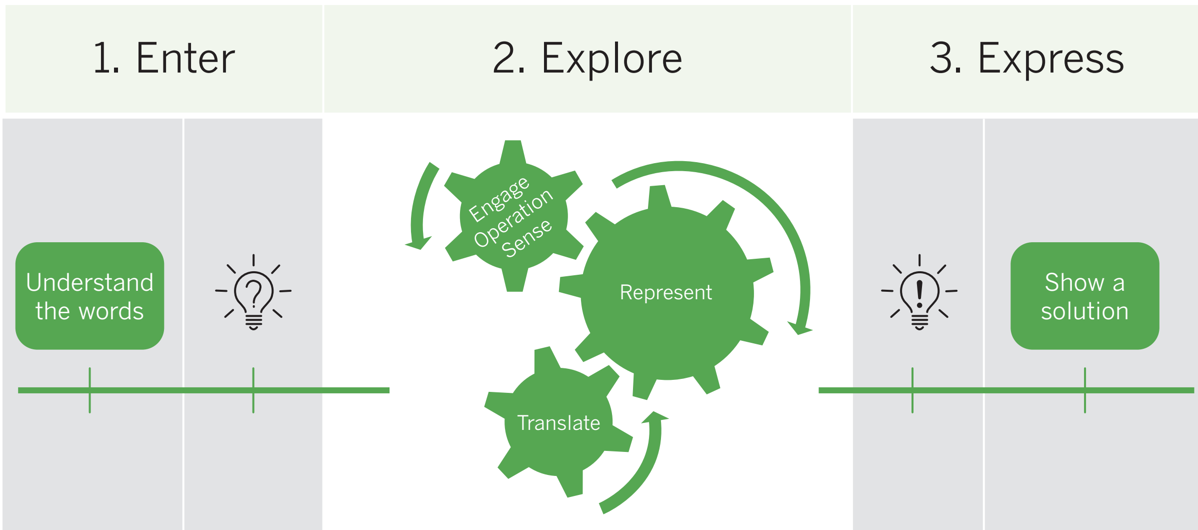
To begin, read the problem in Figure 2.1. Don't try to solve it just yet. Instead, put yourself in the place of your students (who might be listening to you read the problem), and as you enter the problem, focus on understanding the words in it. Try using your own words to rephrase, without focusing on the quantities. If this is difficult, substitute the quantities with the word *some*. This will help you avoid jumping to the solution path before you fully explore the problem. Look at the model in Figure 2.2 to remind yourself of how these tasks fit into the problem-solving process. Congratulations, you are now ready to enter the mathematizing sandbox!

FIGURE 2.1

Paulo was cleaning up the crayons at his table. He put 9 crayons in the basket. Then Paulo found 6 more crayons under the table and put them in the basket. How many crayons are in the basket now?

ENTER THE PROBLEM

FIGURE 2.2 A MODEL FOR MATHEMATIZING WORD PROBLEMS



To explore, you may need to take notes of your explorations on scratch paper. Once you can answer these questions, you are ready to show and justify your solutions. Include these in the space provided so that you can easily refer back to them. If your solution includes a concrete model, reproduce that as best you can in a drawing. Add any verbal representation of the problem or additional notes on your thinking that are necessary to make your solution clear. *Remember: When the focus is on mathematizing, finding a solution is not the same as finding the answer. A solution is a representation of the problem that reveals how it can be solved. The answer comes after.* This concept will become clearer as you work through problems and explore the work samples of students and teachers throughout the book.

EXPLORE

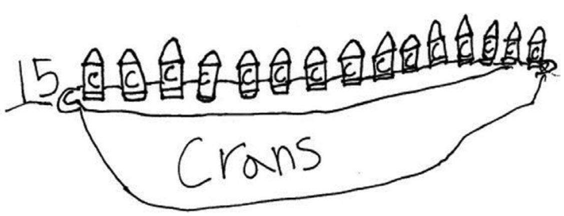
You have taken your first trip to the mathematizing sandbox. You've translated the story of the problem into your own words and explored several different concrete, pictorial, and symbolic representations of the problem situations. You likely have translated your representations into a number sentence that you can then solve, or the answer may have come out of another representation. You have used your operation sense to approach this word problem in a way that reflects deep understanding of the situation rather than simply using computational knowledge.

STUDENTS AND TEACHERS THINK ABOUT THE PROBLEM

Look at the student work in Figures 2.3 and 2.4 and consider how these students describe or draw what is happening in the same word problem.

FIGURE 2.3

STUDENT WORK



"I counted 15 crayons."

TEACHER RESPONSE

I can see the number 15, and 15 crayons in a basket, but I'm not sure if this drawing represents just her answer or her understanding of what happened in the problem. There is no way to know where the 6 is or where the 9 is.

VIDEO

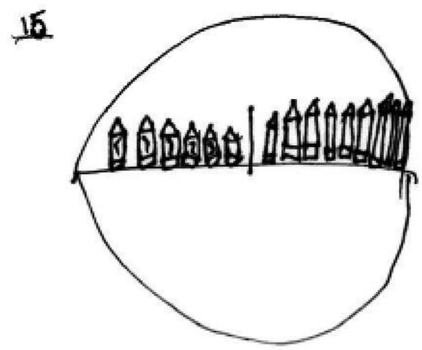
Video 2.1

Cleaning Up Crayons With an Answer

To read a QR code, you must have a smartphone or tablet with a camera. We recommend that you download a QR code reader app that is made specifically for your phone or tablet brand.

FIGURE 2.4

STUDENT WORK



"First I counted 6 and then I counted 9 more."

TEACHER RESPONSE

I see that she has drawn a set of 6 crayons and a set of 9 crayons. Both addends are here. I think I will ask her why she drew 6 crayons first rather than 9 crayons, mostly because the story discusses 9 crayons to start with, and 6 added on. Maybe she made a group of 6 crayons first because it was the last number she heard.

VIDEO

Video 2.2

Cleaning Up Crayons With a Counting-On Strategy

Look at the teacher commentary that follows the student work and consider what the teachers noticed about it. There is also a video available showing each student's solution to the problem, which you may want to watch once you have read through the student work and teacher comments.

What do you notice? Both students give 15 as an answer to the problem and show 15 crayons. In Figure 2.3, the crayons in the basket are presented in a line, but that is the only indication that the student understands this problem as an **Add-To** situation. By contrast, the work in Figure 2.4 appears to be showing both addends, separated by a line, as well as a sum of 15. We want to highlight the distinction between the skill of counting and the skill of adding. In both of these cases, the students likely know how to count to 15 effectively and may even recognize the sum of 9 and 6 as a known fact, but the pictorial representations they each create are different in important ways. As we explore the difference between adding and counting, we will explore how problem solving can differ from computation.

Add-To: A problem situation that includes action happening in the problem. Some quantity is being added to the original quantity.

THE DEVELOPMENT OF COUNTING

Counting a collection of objects is a process of matching words to objects and, after tagging the last object, arriving at a value that represents the whole set. Often, pre-K and kindergarten students will demonstrate some components of the skill of counting but not always the whole coordinated procedure. Because counting is a precursor to addition, it's important to first understand the underlying principles. There are five principles to effective counting (Gelman & Gallistel, 1978): one-to-one correspondence, stable order, cardinality, abstraction, and order irrelevance. Let's look at each one in turn.

ONE-TO-ONE CORRESPONDENCE PRINCIPLE When counting, the child who has the coordinated skills of one-to-one correspondence matches one number word to one object in the collection at a time and tags it. Matching a number word to an object is called *tagging*. Children who have proficiency counting may move each object to one side to show that they have counted it or they may turn it over. If they're counting on a page, they may mark it or cross it out. Distinguishing this way, between counted and uncounted objects, is called *partitioning*. Children who are still developing one-to-one correspondence may point to empty spaces between objects or wave their hands indiscriminately over the collection while saying number words. These actions show neither accurate tagging nor partitioning. Providing students a structure like a counting mat with designated areas for both tagged and not-yet-tagged objects may help them organize their counting. You can also guide their practice by helping them touch, name, and move each object one at a time. In the following example, Maria does not yet demonstrate one-to-one correspondence:

> *A set of nine blocks sits in front of Maria, arranged in a 3 × 3 array. Maria begins to count. She taps one block with her right hand and says, "One." She taps another and says, "Two." Her left hand comes up and points to a different block, and she says, "Three." She then begins to wave her hands over the collection of blocks while racing through the number sequence, "Four, five, six, seven!"*

STABLE ORDER PRINCIPLE When children tag the objects they are counting, the words they say have to come out in the same order each time. Perhaps this is obvious, but children cannot effectively count higher than their stable order of counting sequence. This is the reason why most kindergarten standards include a standard for counting out loud up to a certain number—for example, 100—but the standard for counting concrete objects may only go up to 20. The counting sequence must be stable before it can be used to actively count. Children can also have a stable order in multiple languages. In the following example, Raul is counting cookies:

Six cookies sit on the table in front of Raul. He begins to count, saying, "One, two, three, six, seven, eight!" The teacher responds with, "Could you count them again, Raul?" He responds, "One, two, three, five, six, . . . eight!" The teacher then responds with, "¿Raul, puedes contar esto?" [Can you count this?] He counts, "¡Uno, dos, tres, quatro, cinco, seis!" [One, two, three, four, five, six!] Raul repeats his count accurately. Based on this vignette, Raul may have a stable order of numbers in Spanish but not yet in English.

CARDINALITY PRINCIPLE The principles described so far are about matching one object to one number word. The cardinal number of a set instead does two things. First, it matches the last object to a counting word. Second, it describes a property of the whole collection of objects by answering the question, "How many are there?" Students who don't yet understand the cardinality of the set don't know the answer to this question, even after they have finished counting. They return to the collection and do the count again, or they keep on counting. It takes experience for children to recognize that the last tag refers to a property of the whole set as well as to the count of the last tagged item. Consider the following example:

Four plastic dinosaurs sit on a table in front of Brian. The teacher asks him to count the dinosaurs. He taps one and says, "One." He taps the next three, one at a time, and says, "Two, three, four." The teacher asks, "How many dinosaurs are there, Brian?" Brian responds, "Five, four, three, two, one!" while counting on his fingers. Brian may have accurately counted four dinosaurs, but we aren't sure that he knows that "Four" also represents the quantity of dinosaurs.

These first three counting principles describe *how* to count efficiently. In some ways these skills are similar to other procedural skills students learn. The last two principles of counting extend the *hows* of counting to the *whats*.

Quantity: A number, including its unit of measure (Kelemanik, Lucenta, & Creighton, 2016).

ABSTRACTION PRINCIPLE What is countable? People are countable, but are imaginations or dreams countable? Does a collection of four elephants have the same numerosity (or cardinality) as four pennies? It takes children some time to recognize categories that are countable and to recognize that "4" can represent four large objects or four small objects, four concrete or four abstract objects, and still have the same numerosity of "4." For this reason, we emphasize the importance of naming the unit we are counting. The word ***quantity*** is chosen intentionally in this book because it represents both ideas—numerosity and the unit being counted. For example, the student whose work is featured in Figure 2.3 identifies 15 "crans" (crayons), but the student whose work is shown in Figure 2.4 does not mention the word at all. The teacher made the assumption that the child is discussing crayons (likely based on the drawing), but it is just an assumption that should be regularly confirmed with a question: "You have 15 what?" An interesting answer might be "15 students." How does this change your interpretation of the student's understanding of the problem? As students count sets of objects, continue to ask what they are counting.

Commutative property of addition: The property of addition that states that two quantities can be added in either order without changing the sum.

ORDER IRRELEVANCE PRINCIPLE The most important idea about the order irrelevance principle is that objects in a collection can be counted in any order, as long as each is counted exactly once. In some ways, order irrelevance is like the **commutative property of addition**, an idea that students will learn to understand in grades K–2 but may not name until later. Challenge students to change which object is the first to be counted. Another, more challenging task is to ask students to count a collection of objects and tell them which one should be the last one counted. For example, "Count these bears any way you want, but I want you to count the red one last."

Understanding the basic principles of the counting procedure highlights the conceptual ideas that students must make sense of and also the skills they must acquire to count efficiently. These are skills that generally precede learning the action of addition. It's important to recognize these early ideas about counting as you listen to students working. Observe their counting, even as they enter second grade; and when students make errors in their addition and subtraction, consider assessing whether or not their counting skills are responsible for those errors. As you explore addition with your students, you may realize they are missing one or more of these counting principles. Look for opportunities to pause and address the counting skills that may still be developing when you encounter a concern. The focus of this book is on understanding the problem contexts that underlie most word problems and the operations that represent them. In this chapter we will identify important counting skills and then begin to apply them to word problems that can be represented with addition or subtraction.

COUNTING OR ADD-TO?

Earlier in this chapter we briefly introduced the Add-To problem situation with the student work in Figure 2.4. This book is focused on developing students' operation sense and strengthening their ideas of what an operator like addition or subtraction can do within the context of a problem situation. Because of this focus, we highlight the difference between the context of the problem and the computation that it takes to solve it. Counting is an algorithm or procedure, in some ways like adding double-digit numbers. There are concepts to understand, and the goal is also for students to count efficiently.

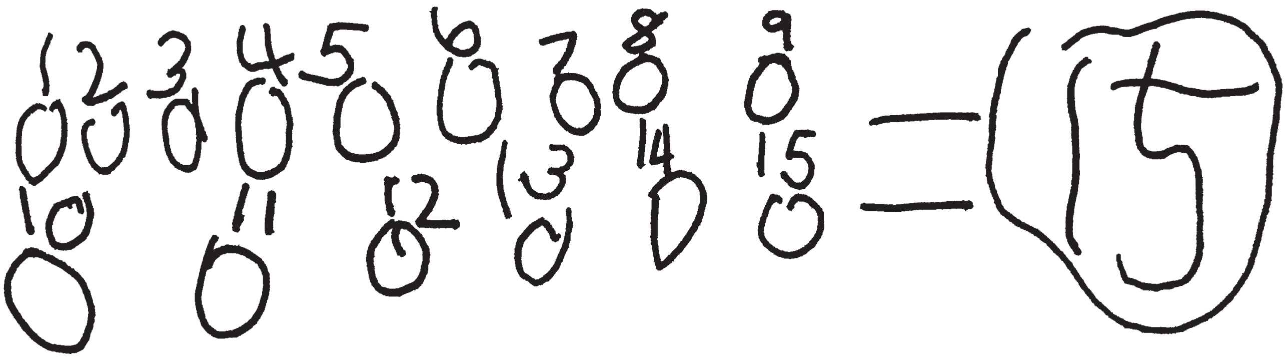
Try focusing on the problem situation in Figure 2.1 and read it like a story. Paulo has some crayons (start). Paulo finds more crayons (change). Paulo now has a new quantity of crayons (result). Any one of these quantities could be missing in a word problem, but in this case we have to find the result, or how many crayons Paulo has at the end of the story.

When problem solving, we want students to act out what happens in a story. Fortunately, young children naturally focus on the narrative when they solve word problems (Kilpatrick, Swafford, & Findell, 2001), but as students get older they tend to do this less because our efforts to teach them to do calculations may turn their attention away from what is happening in the problem. When working with young children, this is a great opportunity to capitalize on what they naturally want to do—act it out! This will help them build good problem-solving habits early.

Return to the student work in Figure 2.4. One might say that it shows a true representation of an Add-To problem because it starts with 6 crayons and adds 9 more. But we would argue that this may not be the case. In the problem situation, Paulo starts with a set of 9 crayons and adds an additional set of 6 crayons to the basket. The student work in Figure 2.4 appears to begin with a set of 6 crayons and then adds 9 crayons. The sum is the same, so you might say it doesn't matter. But it might. Perhaps the student started drawing crayons from the right side, in which case she started with 9 crayons, as the story requires, but the scrunched-up crayon drawings on the right make that unlikely. Maybe she has a robust sense of the order irrelevance principle or of the commutative property and knows that the answer will be the same no matter in what order the addends are put together. The teacher has an entirely different theory about the student's choice to start with 6. Without the opportunity to ask or observe, it is unclear why the student started with 6 crayons, and without learning more from the student, we don't know if she sees this situation accurately (as written), even though her answer is correct.

The straight line of 15 crayons in Figure 2.3 shares a different picture. It is possible that the student counted to find 15, rather than actively joining two sets of crayons. Maybe the student already knew the fact $9 + 6 = 15$

FIGURE 2.5



and therefore did not act out the problem but instead represented only the sum of 15. It is also possible that the student drew 9 crayons and then drew 6 more crayons before going back to show the cardinality of the set of 15; however, nothing in the drawing indicates a separation between the addends. We would need more information to know what the student was thinking, but there is little evidence that she is modeling two different addends in this representation. In a lesson focused on problem solving, this difference matters greatly because we are interested in how students are making sense of the problem situation.

The student work shown in Figure 2.5 better shows the action of the story. The problem situation starts with 9 crayons, and 6 more are added (the computation shows a counting-on strategy, sometimes also called adding on or counting up). The quantity 9 is the start, adding 6 is the change, and the result is shown as 15 crayons. Maybe you acted this out using counters when you solved the problem at the beginning of the chapter, which is appropriate because Add-To problem situations are easily represented with action.

If we want students to understand what is happening in a problem situation and to use their tools to accurately act out, draw, or otherwise model that situation, we have to pay the most attention to this process. Sometimes students show us efficient calculation strategies and can get to correct answers—which are also important accomplishments—but does that always show they understand the situation? As we focus our attention here on the actions and relationships in word problems, we might look differently at a student sample like the one in Figure 2.6. Does this student understand the problem situation? Or not? How could we know? There is evidence that the student understands this fact family, but this work sample does not reveal information about how the student made sense of the word problem itself.

FIGURE 2.6

9+6=15
6+9=15
15=6+9
15=9+6